No-Go Theorems in Noncommutative Quantum Mechanics

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Todas as correções determinadas pelo júri, e só essas, foram efetuadas.

O Presidente do Júri,

Porto, ______/______/_________

"We all make choices in life, but in the end our choices make us."

-Andrew Ryan

"If I have seen further it is by standing on the shoulders of giants."

-Sir Isaac Newton

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Resumo

Neste trabalho são abordados teoremas de veto ("no-go") no contexto de Mecânica Quântica Não-Comutativa. O objectivo principal é verificar se teoremas como o Teorema de Não-Clonagem mantêm-se válidos quando considerados no Espaço de Fase Não-Comutativo genérico, teoremas estes que são importantes no contexto da Teoria da Informação Quântica. Será feito um pequeno resumo da formulação de Wigner-Weyl da Mecânica Quântica, seguido de uma discussão do teorema de Não-Clonagem no Espaço de Fase habitual, assim como uma discussão da sua generalização. Por fim, é provado que estes teoremas continuam válidos em Espaços de Fase Não-Comutativos.

Abstract

In this work, No-Go Theorems in the context of Non-Commutative Quantum Mechanics are addressed. The main focus is to see whether theorems such as the No-Cloning Theorem still hold when a generic Non-Commutative Phase Space is considered. This is of great importance, for instance, in the context of Quantum Information Theory. A brief summary of Wigner-Weyl formulation of Quantum Mechanics is given, followed by the discussion of the No-Cloning Theorem in the standard Phase Space, as well as a discussion of its generalization. Finally, it is proven that theorems of this type hold on a Non-Commutative Phase Space.

Contents

Chapter 1

Introduction

1.1 No-Go Theorems and Quantum Information

During the inception and development of Quantum Mechanics (QM), Quantum Information posed a significant problem for Information Theorists. The fact that the act of measurement changes the system in analysis meant that classical procedures for information treatment were no longer able to be applied to Quantum Information. This led to the study of quantum systems and information and the subsequent development of theorems that restricted the actions upon quantum states.

Theorems such as the No-Cloning Theorem (which ensured that no random state can be duplicated [1, 2]), the No-Deleting Theorem (which ensured that given two copies of a state, there is no way to delete one of them [3]), imply that classical error correction techniques are useless. For example, it is impossible that, during a quantum computation, a duplicate of a state is created and used for correcting errors. This is vital for practical quantum computing, and for a time it was thought to be a key limitation.

Fortunately, with the advent of first quantum error correcting codes, in 1995, which circumvented the No-Cloning Theorem, Quantum Computation has seen a sharp increase in interest, with the first solid-state quantum processor being created by researchers at Yale University in 2009. Since then, Quantum Computing, albeit still in its early years, is becoming a major focus of study.

These types of theorems are known as No-Go Theorems and some of them, such as the No-Cloning, are the object of study of this work. However, instead of working in the usual QM Phase-Space, the focus will be the study on Non-Commutative Phase Spaces.

1.2 Non-Commutative Phase Space Quantum Mechanics

Non-Commutative Quantum Mechanics (NCQM) is an extension of Quantum Mechanics with commutation relations that are a deformation of the standard Heisenberg-Weyl algebra. The replacement of the algebra

$$[q_i, q_j] = 0, \qquad [p_i, p_j] = 0, \qquad [q_i, p_j] = i\hbar\delta_{ij}, \tag{1.1}$$

with the algebra

$$[q_i, q_j] = i\theta_{ij}, \qquad [p_i, p_j] = i\eta_{ij}, \qquad [q_i, p_j] = i\hbar\delta_{ij}, \tag{1.2}$$

where θ and η are real anti-symmetric matrices and

$$\hbar' = \hbar\left(1 + \frac{\theta\eta}{\hbar^2}\right), \tag{1.3}$$

induces a correlation between space directions and momentum directions, or even a quantization of both configuration and momentum space.

Non-Commutativity has recently gained interest due to string theory, as the dynamics of strings can be described by a gauge theory in a non-commutative space (see, for example, Refs. [4, 5]). Since standard QM is the low-energy, finite number of particles limit of other fundamental theories, Non-Commutativity might appear as a small effect at the quantum mechanical level. Because of this, a great amount of work has been devoted to Non-Commutativity, including topics such as: Non-Commutative Geometry (see for example Ref. [6]); the appearance of Non-Commutativity in particles affected by magnetic fields; Non-Commutativity in Quantum Field Theory (see for example Ref. [7]); particles in well studied potentials, but in a NC Phase Space, such as a particle in a central potential (see for example Ref. [8]), the Gravitational Quantum Well (see for example Ref. [9, 10, 11]), the Harmonic Oscillator (see for example Ref. [12]) and the Hydrogen Atom; applications to cosmology have also been considered (see for example Refs. [10, 13]) as well as other works.

A crucial part of these works has been devoted to creating an alternative formulation of NCQM based on the Weyl-Wigner formulation of quantum mechanics(see for example Refs. [14, 15]), which can be useful in the treatment of these new uncertainty relations (see Refs. [16, 17, 18, 19]).

In this work, we will focus on seeing if Non-Commutativity, using Weyl-Wigner formulation, has any influence in No-Go Theorems such as the No-Cloning Theorem.

Chapter 2

Weyl-Wigner Formalism of Quantum Mechanics

In the same way as classical mechanics has various equivalent formulations, such as Newtonian, Lagrangian and Hamiltonian, QM allows for different formulations. In this section, we will briefly describe the Weyl-Wigner (WW) formulation. This Chapter and the succeeding ones assume the Einstein's summation convention.

2.1 Weyl-Wigner transform and the Wigner function

In the standard formulation of QM, the key object is the wave-function; the core of WW formulation is the Wigner function, which is related to the wavefunction by:

$$f(q_i, p_i) = \int \psi^* \left(\overrightarrow{q} - \frac{\overrightarrow{y}}{2} \right) \psi \left(\overrightarrow{q} + \frac{\overrightarrow{y}}{2} \right) e^{-\frac{i p_i y_i}{\hbar}} d^d y, \tag{2.1}$$

where q_i are positions, p_i are the momenta, $\hbar$ is the reduced Plank constant and d is the number of spacial dimensions.

Another important tool is the Wigner-Weyl transformation (see Ref. [20]), which maps operators in a Hilbert space to functions in phase space and vice-versa. Given an operator $\hat{A}$, the Wigner transform is defined as:

$$W\left(\hat{A}\right)(q_i, p_i) = \int d^d y \left\langle \overrightarrow{q} - \frac{\overrightarrow{y}}{2} | \hat{A} | \overrightarrow{q} + \frac{\overrightarrow{y}}{2} \right\rangle e^{-\frac{i p_i y_i}{\hbar}}. \tag{2.2}$$

Note that the Wigner transform has the properties:

$$W\left(\widehat{q}_i\right) = q_i, \tag{2.3}$$

$$W\left(\widehat{p}_i\right) = p_i, \tag{2.4}$$

$$W\left(Id\right) = 1. \tag{2.5}$$

We can now see that in fact

$$f(q_i, p_i) = W\left(\widehat{\rho}\right) = W\left(|\psi\rangle\langle\psi|\right), \tag{2.6}$$

where $\widehat{\rho} = |\psi\rangle\langle\psi|$ is the density matrix associated with a state $|\psi\rangle$.

This mapping is one-to-one and admits an inverse, the Weyl transform:

$$W^{-1}\left(g\right) = \int \frac{d^{2d}k}{\left(2\pi\right)^{2d}} \int d^{2d}z\, g \cdot e^{ik_i\hat{z}^i} e^{-ik_i z^i}, \tag{2.7}$$

where we used $z = (q_i, p_i)$ for the coordinates in the phase space, and $\widehat{z} = (\widehat{q}_i, \widehat{p}_i)$ are the position and momenta operators in Hilbert space, and $2d$ is the dimension of the phase space.

This means that, for any operator $\hat{A}$ (see Ref. [7]):

$$\hat{A} = W^{-1}\left(W\left(\hat{A}\right)\right) = \int \frac{d^{2d}k}{\left(2\pi\right)^{2d}} \int d^{2d}z\, W\left(\hat{A}\right) \cdot e^{ik_i\hat{z}^i} e^{-ik_i z^i}. \tag{2.8}$$

Note that $W\left(\hat{A}\right)$ is a function of $z = (q_i, p_i)$, which are eigenvalues of $\widehat{z} = (\widehat{q}_i, \widehat{p}_i)$, and thus the fact that $e^{ik_i\hat{z}^i}$ forms a complete basis for operators allows these transformations to be one-to-one.

2.2 Moyal product

The last mathematical object to be introduced is the so-called Moyal or star product, defined so that:

$$W\left(\hat{A}\right) \star W\left(\hat{B}\right) = W\left(\hat{A}\hat{B}\right). \tag{2.9}$$

In general, it can be proven that Moyal product has the form (see Refs. [21, 22]):

$$\begin{aligned}
A \star B &= A e^{\frac{i\hbar}{2}\overleftarrow{\partial_{z_i}}\Omega_{ij}\overrightarrow{\partial_{z_j}}} B \\
&= AB + \sum_{n=1}^{\infty} \frac{1}{n!}\left(\frac{i\hbar}{2}\right)^n A \left(\overleftarrow{\partial_{z_i}}\Omega_{ij}\overrightarrow{\partial_{z_j}}\right)^n B \\
&= AB + \sum_{n=1}^{\infty} \frac{(i\hbar)^n}{n!2^n}\left(\partial^{(n)}_{z_{\alpha_1}\dots z_{\alpha_n}} A\right)\left(\partial^{(n)}_{z_{\beta_1}\dots z_{\beta_n}} B\right)\Omega_{\alpha_1\beta_1}\dots\Omega_{\alpha_n\beta_n}, \qquad (2.10)
\end{aligned}$$

where the arrows mean the derivatives are applied to the left or right, and

$$\Omega = \begin{pmatrix} 0 & \mathrm{Id}_{d\times d} \\ -\mathrm{Id}_{d\times d} & 0 \end{pmatrix} \qquad (2.11)$$

is the matrix of commutation relations, i.e., for $\widehat{z} = (\widehat{q}_i, \widehat{p}_i)$,

$$[\widehat{z}_i, \widehat{z}_j] = i\hbar\Omega_{ij}, \qquad (2.12)$$

according to the Heisenberg-Weyl algebra (c. f. eq. (1.1)).

Note that by definition Ω_{ij} is antisymmetric because $[\widehat{z}_i, \widehat{z}_j] = -[\widehat{z}_j, \widehat{z}_i]$.

In addition, taking the Weyl transform of eq. (2.9), we see that:

$$\begin{aligned}
W^{-1}\left(W\left(\hat{A}\right) \star W\left(\hat{B}\right)\right) &= W^{-1}\left(W\left(\hat{A}\hat{B}\right)\right) \\
&= \hat{A}\hat{B}. \qquad (2.13)
\end{aligned}$$

2.3 Weyl-Wigner Formulation of Quantum Mechanics

In the WW formulation of QM, expectation values of operators can be calculated using the Wigner-Weyl transformation, and the Moyal equation[21, 22] accounts for the dynamical evolution, just as the Schrödinger equation for the standard formulation of QM.

Using the definition of the Weyl transform, one can show that the expectation values of operators are given by:

$$\left\langle \widehat{G} \right\rangle = \int f(z) g(z)\, d^{2d}z. \qquad (2.14)$$

Similarly, one can prove that the evolution of the system is described by the Moyal equation:

$$\frac{\partial f}{\partial t} = \frac{H \star f - f \star H}{i\hbar} := \frac{1}{i\hbar} \{H, f\}_\star , \tag{2.15}$$

where $H = W\left(\widehat{H}\right)$ is the phase space version of the Hamiltonian and where we introduced the Moyal brackets defined as:

$$\{A, B\}_\star = A \star B - B \star A. \tag{2.16}$$

Furthermore, one can also prove that for stationary systems this can be expressed as (see Ref. [22]):

$$H(z) \star f(z) = E f(z) . \tag{2.17}$$

2.4 Main Properties

2.4.1 Trace of operators

A useful property is the relation between the trace of an operator and its Wigner transform. The proof is straightforward:

Theorem: For any operator $\hat{A}$,

$$\mathrm{Tr}\left(\hat{A}\right) = \int d^{2d}z\, W\left(\hat{A}\right). \tag{2.18}$$

Proof: Using the definition of the Weyl transform, we have:

$$\begin{aligned} \mathrm{Tr}\left(\hat{A}\right) &= \mathrm{Tr}\left(W^{-1}\left[W\left(\hat{A}\right)\right]\right) \\ &= \mathrm{Tr}\left(\int \frac{d^{2d}k}{(2\pi)^{2d}} \int d^{2d}z\, W\left(\hat{A}\right) e^{ik_i\hat{z}^i} e^{-ik_i z^i}\right) \\ &= \int d^{2d}z\, W\left(\hat{A}\right) \mathrm{Tr}\left(\int \frac{d^{2d}k}{(2\pi)^{2d}} e^{ik_i\hat{z}^i} e^{-ik_i z^i}\right), \end{aligned} \tag{2.19}$$

given that Wigner transform is not an operator, and thus is not affected by a trace. Then, by property of the trace and of the integration in k,

$$\mathrm{Tr}\left(\int \frac{d^{2d}k}{(2\pi)^{2d}}\, e^{ik_i\hat{z}^i} e^{-ik_iz^i}\right) = \int \frac{d^{2d}k}{(2\pi)^{2d}}\, e^{ik_iz^i} e^{-ik_iz^i}$$
$$= \int \frac{d^{2d}k}{(2\pi)^{2d}} = 1, \quad (2.20)$$

where we evaluated the trace in the eigenbasis of $\widehat{z}$ in the first step, and thus

$$\mathrm{Tr}\left(\hat{A}\right) = \int d^{2d}z\, W\left(\hat{A}\right) \mathrm{Tr}\left(\int \frac{d^{2d}k}{(2\pi)^{2d}}\, e^{ik_i\hat{z}^i} e^{-ik_iz^i}\right)$$
$$= \int d^{2d}z\, W\left(\hat{A}\right).$$

2.4.2 Invariance under phase space integration

One of the most important properties that will be used is the invariance under phase space integration of the Moyal Product.

Theorem: For any Ω_{ij} that defines a Moyal Product $\star$, and for any two functions A, B defined in Phase-Space, the following is true:

$$\int d^{2d}z\, A \star B = \int d^{2d}z\, AB. \quad (2.21)$$

Proof: First, one expands the Moyal Product in its series expansion:

$$\int d^{2d}z\, A \star B = \int d^{2d}z\, AB + \sum_{n=1}^{\infty} \frac{1}{n!}\left(\frac{i\hbar}{2}\right)^n \int d^{2d}z \left(A \left(\overleftarrow{\partial_{z_i}}\Omega_{ij}\overrightarrow{\partial_{z_j}}\right)^n B\right). \quad (2.22)$$

Now one has only to prove that the integrals in the second term all yield zero. This is done by expanding $\left(\overleftarrow{\partial_{z_i}}\Omega_{ij}\overrightarrow{\partial_{z_j}}\right)^n$ into individual terms (cf. eq. 2.10) and then transforming the integrand into a total derivative minus a term that is symmetric in the change of a particular pair of indexes i, j for which there is an associated Ω_{ij} matrix element, and since the Ω matrix is antisymmetric, that term yields zero (notice that one assumes that A, B and their derivatives vanish at infinity):

$$\left(\partial^{(n)}_{z_{\alpha_1}\ldots z_{\alpha_n}} A\right)\left(\partial^{(n)}_{z_{\beta_1}\ldots z_{\beta_n}} B\right)\Omega_{\alpha_1\beta_1}\ldots\Omega_{\alpha_n\beta_n} =$$

$$\partial_{z_{\alpha_1}}\left(\left(\partial^{(n-1)}_{z_{\alpha_2}\dots z_{\alpha_n}}A\right)\left(\partial^{(n)}_{z_{\beta_1}\dots z_{\beta_n}}B\right)\right)\Omega_{\alpha_1\beta_1}\dots\Omega_{\alpha_n\beta_n}-$$

$$-\left(\partial^{(n)}_{z_{\alpha_1}\dots z_{\alpha_n}}A\right)\left(\partial^{(n+1)}_{z_{\alpha_1}z_{\beta_1}\dots z_{\beta_n}}B\right)\Omega_{\alpha_1\beta_1}\dots\Omega_{\alpha_n\beta_n}. \tag{2.23}$$

Since the first term is a total derivative,

$$\int d^{2d}z\,\partial_{z_{\alpha_1}}\left(\left(\partial^{(n-1)}_{z_{\alpha_2}\dots z_{\alpha_n}}A\right)\left(\partial^{(n)}_{z_{\beta_1}\dots z_{\beta_n}}B\right)\right)\Omega_{\alpha_1\beta_1}\dots\Omega_{\alpha_n\beta_n}=0. \tag{2.24}$$

Because $\partial_{z_{\alpha_1}}\partial_{z_{\beta_1}} = \partial_{z_{\beta_1}}\partial_{z_{\alpha_1}}$ and $\Omega_{z_{\alpha_1}z_{\beta_1}}$ is antisymmetric in the exchange of indexes $\alpha_1 \leftrightarrow \beta_1$,

$$\left(\partial^{(n)}_{z_{\alpha_1}\dots z_{\alpha_n}}A\right)\left(\partial^{(n+1)}_{z_{\alpha_1}z_{\beta_1}\dots z_{\beta_n}}B\right)\Omega_{\alpha_1\beta_1}\dots\Omega_{\alpha_n\beta_n}=0, \tag{2.25}$$

and therefore

$$\sum_{n=1}^{\infty}\frac{1}{n!}\left(\frac{i\hbar}{2}\right)^n\int d^{2d}z\,\left(A\left(\overleftarrow{\partial_{z_i}}\Omega_{ij}\overrightarrow{\partial_{z_j}}\right)^n B\right)=0, \tag{2.26}$$

which means

$$\int d^{2d}z\,A\star B=\int d^{2d}z\,AB. \tag{2.27}$$

Chapter 3

Weyl-Wigner Formalism of Non-Commutative Quantum Mechanics

3.1 Non-Commutative Wigner Transform

In order to describe Non-Commutative Quantum Mechanics (NCQM) in Phase Space, one needs a proper way to map NCQM operators in Hilbert Space into functions in $\mathbb{R}^{2d}$.

This means finding a one-to-one linear map V so that:

1. $V\left(Id\right) = 1$,
2. $V\left(\widehat{q}\right) = q$,
3. $V\left(\widehat{p}\right) = p$,
4. $V\left(\hat{A}\widehat{B}\right) = V\left(\hat{A}\right) \star_{NC} V\left(\widehat{B}\right)$,

for a phase space with deformed commutation relations of the usual Heisenberg-Weyl algebra (see eq. (1.2)):

$$[q_i, q_j] = i\theta_{ij}, \qquad [p_i, p_j] = i\eta_{ij}, \qquad [q_i, p_j] = i\hbar'\delta_{ij}, \tag{3.1}$$

where θ and η are real anti-symmetric matrices.

However, this map is not unique, and thus there are several proposals for this map. In Ref. [14] this issue is discussed and different maps are compared. It is argued that a suitable map is the following:

$$W_{NC}\left(\hat{A}\right)(z) = h^{-d}\int d^d y\, d^d x\, e^{-i\Pi(z)\cdot y}\delta\left(x - R(z)\right)\left\langle x + \frac{\hbar}{2}y|\hat{A}|x - \frac{\hbar}{2}y\right\rangle_{\widehat{R}}, \tag{3.2}$$

where x, y are positions, $R(z), \Pi(z)$ are canonically conjugated variables that are related to z via a Darboux transformation, to be described in the next section (see Ref. [23] for a detailed explanation). It is also argued that this map has exactly the same properties as in section 2.4, i.e.:

$$W_{NC}\left(\hat{A}\widehat{B}\right) = W_{NC}\left(\hat{A}\right)\star_{NC} W_{NC}\left(\widehat{B}\right), \tag{3.3}$$

$$\mathrm{Tr}\left(\hat{A}\right) = \int d^{2d}z\, W_{NC}\left(\hat{A}\right), \tag{3.4}$$

$$\int d^{2d}z\, A\star_{NC} B = \int d^{2d}z\, AB, \tag{3.5}$$

with

$$A\star_{NC} B = Ae^{\frac{i\hbar'}{2}\overleftarrow{\partial_{z_i}}\Omega_{ij}^{NC}\overrightarrow{\partial_{z_j}}}B, \tag{3.6}$$

where the Ω matrix is now

$$\Omega^{NC} = \begin{pmatrix} \frac{1}{\hbar}\Theta & \mathrm{Id}_{d\times d} \\ -\mathrm{Id}_{d\times d} & \frac{1}{\hbar}N \end{pmatrix}, \tag{3.7}$$

and where

$$\Theta = (\theta_{ij}), \qquad N = (\eta_{ij}) \tag{3.8}$$

are the matrices of the commutation relations elements.

These maps allows for NCQM to be described in phase-space just as QM. One now needs a way to connect variables in NCQM and variables in QM.

3.2 Darboux transformation

The Darboux transformation, or Seiberg-Witten map, is a non-canonical linear transformation between the two sets of phase space variables with different commutation relations, usually between commutative variables that obey the Heisenberg-Weyl algebra and non-commutative variables that obey the algebra shown above, and that we repeat for convenience:

$$[q_i, q_j] = i\theta_{ij}, \tag{3.9}$$

$$[p_i, p_j] = i\eta_{ij}, \tag{3.10}$$

$$[q_i, p_j] = i\hbar'\delta_{ij}. \tag{3.11}$$

The aim of this section is to work out this transformation, establish its properties in order to specify a link between variables in NCQM with the ones in QM.

In order to relate commutative and non-commutative variables, consider a linear transformation to change the variables $z = (q_i, p_i)$ that obey the above commutation relations into standard commutative variables $z^C = \left(q_i^C, p_i^C\right)$ that satisfy the Heisenberg-Weyl commutation relations:

$$q_i = A_{ij}q_j^C + B_{ij}p_j^C, \tag{3.12}$$

$$p_i = C_{ij}q_j^C + D_{ij}p_j^C. \tag{3.13}$$

Theorem (see Ref. [14]): The matrices A, B, C and D obey the relationships:

$$AD^T - BC^T = \frac{\hbar'}{\hbar}\mathrm{Id}_{d\times d}, \tag{3.14}$$

$$AB^T - BA^T = \frac{1}{\hbar}\Theta, \tag{3.15}$$

$$CD^T - DC^T = \frac{1}{\hbar}N, \tag{3.16}$$

where

$$\Theta = (\theta_{ij}), \qquad N = (\eta_{ij}) \tag{3.17}$$

are the matrices of the commutation relations elements for position and momenta, respectively, for the non-commutative variables.

Proof of eq. (3.14):

$$\begin{aligned}
i\theta_{ij} &= [q_i, q_j] \\
&= A_{ik}A_{jl}\left[q_k^C, q_l^C\right] + A_{ik}B_{jl}\left[q_k^C, p_l^C\right] \\
&\quad -B_{ik}A_{jl}\left[q_l^C, p_k^C\right] + B_{ik}B_{jl}\left[p_k^C, p_l^C\right] \\
&= i\hbar\left(A_{ik}B_{jl}\delta_{kl} - B_{ik}A_{jl}\delta_{kl}\right) \\
&= i\hbar\left(A_{ik}B_{kj}^T - B_{ik}A_{kj}^T\right) \\
&= i\hbar\left(AB^T - BA^T\right)_{ij}.
\end{aligned} \tag{3.18}$$

Proof of eq. (3.15):

$$\begin{aligned}
i\eta_{ij} &= [p_i, p_j] \\
&= C_{ik}C_{jl}\left[q_k^C, q_l^C\right] + C_{ik}D_{jl}\left[q_k^C, p_l^C\right] \\
&\quad -D_{ik}C_{jl}\left[q_l^C, p_k^C\right] + D_{ik}D_{jl}\left[p_k^C, p_l^C\right] \\
&= i\hbar\left(C_{ik}D_{jl}\delta_{kl} - D_{ik}C_{jl}\delta_{kl}\right) \\
&= i\hbar\left(C_{ik}D_{kj}^T - D_{ik}C_{kj}^T\right) \\
&= i\hbar\left(CD^T - DC^T\right)_{ij}.
\end{aligned} \tag{3.19}$$

Proof of eq. (3.16):

$$\begin{aligned}
i\hbar'\delta_{ij} &= [q_i, p_j] \\
&= A_{ik}C_{jl}\left[q_k^C, q_l^C\right] + A_{ik}D_{jl}\left[q_k^C, p_l^C\right] \\
&\quad -B_{ik}C_{jl}\left[q_l^C, p_k^C\right] + B_{ik}D_{jl}\left[p_k^C, p_l^C\right] \\
&= i\hbar\left(A_{ik}D_{jl}\delta_{kl} - B_{ik}C_{jl}\delta_{kl}\right) \\
&= i\hbar\left(A_{ik}D_{kj}^T - B_{ik}C_{kj}^T\right) \\
&= i\hbar\left(AD^T - BC^T\right)_{ij}.
\end{aligned} \tag{3.20}$$

Note, however, that these relations are not enough to fully determinate the matrices Θ and N, as there are $4d^2$ parameters and $\frac{d}{2}(3d-1)$ independent equations, and thus there are $\frac{d}{2}(5d+1)$ free parameters.

One can, however, consider simplifications. For instance, without loss of generality, we can take A and D to be the identity matrix. We then have:

$$-BC^T = \frac{\hbar'}{\hbar}\mathrm{Id}_{d\times d}, \tag{3.21}$$

$$B^T - B = \frac{1}{\hbar}\Theta, \tag{3.22}$$

$$C - C^T = \frac{1}{\hbar}N. \tag{3.23}$$

The most commonly considered case has $\Theta = \theta\epsilon_{ij}$ and $N = \eta\epsilon_{ij}$, where $\epsilon_{ij} = \epsilon_{ijk}$ with $k \neq i, j$ is antisymmetric in i, j. Then, we can take:

$$B = \frac{\Theta}{2\hbar}, \qquad\qquad C = -\frac{N}{2\hbar}. \tag{3.24}$$

Thus, we then end up with the following Darboux transformation:

$$q_i = q_i^C + \frac{\theta}{2\hbar}\epsilon_{ij}p_j^C, \tag{3.25}$$

$$p_i = p_i^C - \frac{\eta}{2\hbar}\epsilon_{ij}q_j^C. \tag{3.26}$$

Finally, note that $\hbar'$ (c.f. eqs. (3.11) and (3.21)) and $\hbar$ are thus related by:

$$\hbar' = \hbar\left(1 + \frac{\theta\eta}{4\hbar^2}\right). \tag{3.27}$$

Chapter 4

No-Go Theorems in Quantum Mechanics

In this Chapter we provide an overview of No-Go Theorems in QM. First we start by proving the No-Cloning Theorem in the context of QM, and then provide a generalization for No-Go Theorems that include the No-Cloning and No-Deleting Theorem as special cases.

4.1 No-Cloning in Quantum Mechanics

The concept of cloning is a simple one. The idea is to take a generic system and an empty one, and evolve them so that one ends up with two copies of the original system. However, as we will see next, assuming cloning for a generic state restricts the states that can be cloned, and thus cloning cannot be done in generality.

Theorem (No-Cloning in QM): Let $|\psi\rangle$ be a generic quantum state and let $|0\rangle$ be an empty state. Then it is not possible to evolve these two states into two copies of $|\psi\rangle$ for any quantum state $|\psi\rangle$.

Proof: The classical proof of the theorem is done by reductio ad absurdum.

Assume cloning is possible. Then there is an Hamiltonian, $\widehat{H}$, with an associated unitary time evolution operator, $\widehat{U} = e^{\frac{i}{\hbar}\int \widehat{H} dt}$, so that

$$\widehat{U}\, |\psi\rangle_A\, |0\rangle_B = |\psi\rangle_A\, |\psi\rangle_B\,, \tag{4.1}$$

for any given unknown state $|\psi\rangle$ and an empty state $|0\rangle$.

Then, if $|\phi\rangle$ is another state, i.e.,

$$\widehat{U}\,|\phi\rangle_A\,|0\rangle_B = |\phi\rangle_A\,|\phi\rangle_B\,, \tag{4.2}$$

one has:

$$\begin{aligned}\langle\phi|\psi\rangle &= \langle\phi|\psi\rangle_A\,\langle 0|0\rangle_B \\ &= \left(\langle\phi|_A\,\langle 0|_B\right)\left(|\psi\rangle_A\,|0\rangle_B\right) \\ &= \left(\langle\phi|_A\,\langle 0|_B\right)\widehat{U}^\dagger\widehat{U}\left(|\psi\rangle_A\,|0\rangle_B\right) \\ &= \left(\langle\phi|_A\,\langle\phi|_B\right)\left(|\psi\rangle_A\,|\psi\rangle_B\right) \\ &= \langle\phi|\psi\rangle^2\,,\end{aligned} \tag{4.3}$$

where it was used that $\widehat{U}^\dagger\widehat{U} = Id$ and $\langle 0|0\rangle = 1$.

Thus, $\langle\phi|\psi\rangle = 0$ or $\langle\phi|\psi\rangle = 1$, which cannot be true for all states$|\phi\rangle$ and $|\psi\rangle$. The contradiction arises from the fact that we assumed that cloning was possible for any given state. Hence, there can be no cloning.

Note that a similar proof can be performed in reversed order, i.e. there is no time evolution operator $\widehat{U}'$ so that:

$$\widehat{U}'\,|\psi\rangle_A\,|\psi\rangle_B = |\psi\rangle_A\,|0\rangle_B \tag{4.4}$$

for any generic quantum state $|\psi\rangle$, this being the No-Deleting Theorem. This can be viewed as time-reversed dual of the No-Cloning Theorem, as by definition of time evolution operator

$$\widehat{U}\,(t_0,t_1)\,\widehat{U}\,(t_1,t_0) = \mathrm{Id}, \tag{4.5}$$

and thus

$$\widehat{U}\,(t_0,t_1)\,|\psi\rangle_A\,|0\rangle_B = |\psi\rangle_A\,|\psi\rangle_B \Longleftrightarrow \widehat{U}\,(t_1,t_0)\,|\psi\rangle_A\,|\psi\rangle_B = |\psi\rangle_A\,|0\rangle_B\,. \tag{4.6}$$

4.2 No-Go Generalization in Quantum Mechanics

In this section, we will introduce a theorem stronger than the No-Cloning and No-Deleting Theorems, and which includes both as special cases. The key result of the theorem is that one cannot take one or more copies of quantum systems and partially superpose them with a fixed state. This proof is similar to one given in Ref. [24].

Theorem: There is no Hamiltonian, $\widehat{H}$, with an associated time evolution operator, $\widehat{U}$, so that for a fixed state $|\phi\rangle$ and for any state $|\psi\rangle$ the following is true:

$$\widehat{U}\,|\psi\rangle^{\otimes k}\,|0\rangle^{\otimes N-k} = |\varphi\rangle^{\otimes n}\,|0\rangle^{\otimes N-n}\,, \tag{4.7}$$

where $|\varphi\rangle = \alpha\,|\psi\rangle + \beta\,|\phi\rangle$, with $|\alpha|^2 + |\beta|^2 = 1$, and where we used the notation:

$$|\psi\rangle^{\otimes k} = \underbrace{|\psi\rangle \otimes \ldots \otimes |\psi\rangle}_{k\,times}. \tag{4.8}$$

Proof: First, note that when $\beta = 0$ (i.e. there is no superposition with another state), if $k < n$, we have the No-Cloning Theorem, and if $k > n$, we have the No-Deleting Theorem.

Thus, we only need to prove the case $0 < \beta < 1$. The goal is to show that this implies a contradiction, i.e. the reductio ad absurdum method.

Suppose there is an Hamiltonian, $\widehat{H}$, so that

$$\widehat{U}\,|\psi\rangle^{\otimes k}\,|0\rangle^{\otimes N-k} = |\varphi\rangle^{\otimes n}\,|0\rangle^{\otimes N-n} \tag{4.9}$$

for any state $|\psi\rangle$ and with $|\varphi\rangle = \alpha\,|\psi\rangle + \beta\,|\phi\rangle$.

Then, if we instead used the state $e^{i\theta}\,|\psi\rangle$, we would have

$$\widehat{U}e^{ik\theta}\,|\psi\rangle^{\otimes k}\,|0\rangle^{\otimes N-k} = |\varphi'\rangle^{\otimes n}\,|0\rangle^{\otimes N-n}\,, \tag{4.10}$$

where $|\varphi'\rangle = \alpha e^{i\theta}\,|\psi\rangle + \beta\,|\phi\rangle$.

However, because $|\psi\rangle \propto e^{i\theta}\,|\psi\rangle$, assuming that states are normalized, we take the hermitic conjugate of eq. (4.9) and multiply it by eq. (4.10) to obtain:

$$e^{ik\theta} = \langle\varphi|\varphi'\rangle^{n}\,. \tag{4.11}$$

Then, by definition of $|\varphi\rangle$ and $|\varphi'\rangle$,

$$\langle\varphi|\varphi'\rangle = e^{i\theta}\,|\alpha|^2 + |\beta|^2\,, \tag{4.12}$$

and thus, because $|\alpha|^2$ and $|\beta|^2$ are real numbers, eq. (4.11) can only be true if $\beta = 0$, which is a contradiction. This concludes the proof.

Chapter 5

Bridging Non-Commutative Quantum Mechanics and Quantum Mechanics

Before approaching the previous theorems in NCQM, one needs a way to relate states in QM with states in NCQM. This chapter intends to show that the transformation

$$\hat{A} \longrightarrow W^{-1}\left(D \circ W_{NC}\left[\hat{A}\right]\right) \tag{5.1}$$

allows for describing states in NCQM through states in QM via a modified Hamiltonian, for a map W_{NC}, and where D is the Darboux transformation for variables in phase-space described in Chapter 3.

5.1 Transforming Operators in NCQM to Operators in QM

As we saw in the previous chapters, the Wigner Transform maps QM operators to phase-space functions, and its inverse, the Weyl transform, maps these phase-space functions again to operators in QM, i.e.

$$W: \mathcal{H}_C \longrightarrow \mathcal{C}\left[\mathbb{R}^{2d}\right], \tag{5.2}$$

$$W^{-1}:\ \mathcal{C}\left[\mathbb{R}^{2d}\right]\longrightarrow\mathcal{H}_C. \tag{5.3}$$

Similarly, any NC Wigner Transform maps NCQM operators to phase-space functions, i.e.

$$W_{NC}:\ \mathcal{H}_{NC}\longrightarrow\mathcal{C}\left[\mathbb{R}^{2d}\right]. \tag{5.4}$$

In order to study states in NCQM, we now intend to use eqs. (5.3) and (5.4) to map operators in NCQM to some other operators in QM by using the fact that W_{NC} yields functions, and W^{-1} has functions as arguments. This means, for an operator $\widehat{A}$, one can define an operator $\widehat{\mathcal{A}}$ defined as

$$\widehat{\mathcal{A}}=W^{-1}\left(W_{NC}\left[\widehat{A}\right]\right) \tag{5.5}$$

in QM. Note that this is simply a mathematical object, and does not necessarily have the same physical meaning as $\hat{A}$.

However, when one considers a product of two operators, $\hat{A}$ and $\widehat{B}$, remembering that (see eqs. (2.9) and (3.3))

$$W_{NC}\left[\widehat{A}\widehat{B}\right]=W_{NC}\left[\widehat{A}\right]\star_{NC}W_{NC}\left[\widehat{B}\right]$$

$$W\left[\widehat{A}\widehat{B}\right]=W\left[\widehat{A}\right]\star_{C}W\left[\widehat{B}\right],$$

we get:

$$\begin{aligned}W^{-1}\left(W_{NC}\left[\widehat{A}\widehat{B}\right]\right) &= W^{-1}\left(W_{NC}\left[\widehat{A}\right]\star_{NC}W_{NC}\left[\widehat{B}\right]\right)\\ &\neq W^{-1}\left(W_{NC}\left[\widehat{A}\right]\right)W^{-1}\left(W_{NC}\left[\widehat{B}\right]\right),\end{aligned} \tag{5.6}$$

since W^{-1} has the property

$$W^{-1}\left(f\star_C g\right)=W^{-1}\left(f\right)W^{-1}\left(g\right) \tag{5.7}$$

for functions f and g.

Therefore, if we find a linear transformation of variables T, i.e

$$z_i'=\left(T\circ z\right)_i=S_{ij}z_j \tag{5.8}$$

and

$$T \circ f(z) = f(z') = f(S_{ij} z_j), \tag{5.9}$$

so that

$$\begin{aligned} &W^{-1}\left(T \circ \left(W_{NC}\left[\hat{A}\right] \star_{NC} W_{NC}\left[\widehat{B}\right]\right)\right) = \\ &= W^{-1}\left(\left(T \circ W_{NC}\left[\hat{A}\right]\right) \star_{C} \left(T \circ W_{NC}\left[\widehat{B}\right]\right)\right), \end{aligned} \tag{5.10}$$

we can define

$$\widehat{\mathcal{O}} = W^{-1}\left(T \circ W_{NC}\left[\widehat{O}\right]\right) \tag{5.11}$$

for any operator $\widehat{O}$ in NCQM, and get

$$\widehat{\mathcal{A}}\widehat{\mathcal{B}} = W^{-1}\left(T \circ W_{NC}\left[\hat{A}\widehat{B}\right]\right). \tag{5.12}$$

The problem now is to calculate this transformation T.

5.2 The change of variables T

Because of eq. (5.10), we need to find a linear transformation of variables T so that for any two phase-space functions f and g

$$T \circ (f(z) \star_{NC} g(z)) = (T \circ f(z)) \star_{C} (T \circ g(z)). \tag{5.13}$$

Expanding the Moyal products (see eqs. (2.10)), we get:

$$T \circ \left(f e^{\frac{i\hbar'}{2}\overleftarrow{\partial_{z_i}}\Omega^{NC}_{ij}\overrightarrow{\partial_{z_j}}} g\right) = (T \circ f)\, e^{\frac{i\hbar}{2}\overleftarrow{\partial_{z_i}}\Omega^{C}_{ij}\overrightarrow{\partial_{z_j}}} (T \circ g), \tag{5.14}$$

where

$$\Omega^{C} = \begin{pmatrix} 0 & \mathrm{Id}_{d\times d} \\ -\mathrm{Id}_{d\times d} & 0 \end{pmatrix}, \qquad \Omega^{NC} = \begin{pmatrix} \frac{1}{\hbar}\Theta & \mathrm{Id}_{d\times d} \\ -\mathrm{Id}_{d\times d} & \frac{1}{\hbar}N \end{pmatrix}, \tag{5.15}$$

as defined previously.

Assuming T is a linear tranformation, similar to what was done for the Darboux transformation, we can write:

$$z_i \longrightarrow z'_i = (T \circ z)_i = S_{ij} z_j, \tag{5.16}$$

with $i, j = 1, ..., 2d$, or, separating positions from momenta,

$$q_i \xrightarrow{T} q'_i = \alpha_{ij} q_j + \beta_{ij} p_j, \tag{5.17}$$

$$p_i \xrightarrow{T} p'_i = \gamma_{ij} q_j + \zeta_{ij} p_j, \tag{5.18}$$

with $i, j = 1, ..., 2d$, where we defined

$$S = (S_{ij}) = \begin{pmatrix} \boldsymbol{\alpha} & \boldsymbol{\beta} \\ \boldsymbol{\gamma} & \boldsymbol{\zeta} \end{pmatrix}, \tag{5.19}$$

where $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\zeta}$ are the matrices with coeficients $\alpha_{ij}, \beta_{ij}, \gamma_{ij}, \zeta_{ij}$, respectively.

Because we can expand the Moyal product (eq. (2.10)) as:

$$f \star g = fg + \sum_{n=1}^{\infty} \frac{(i\hbar)^n}{n! 2^n} \left(\partial^{(n)}_{z_{\alpha_1} \ldots z_{\alpha_n}} f \right) \left(\partial^{(n)}_{z_{\beta_1} \ldots z_{\beta_n}} g \right) \Omega_{\alpha_1 \beta_1} \ldots \Omega_{\alpha_n \beta_n}, \tag{5.20}$$

we need to calculate terms like $\partial_{z_i} f\,(T \circ z)$:

$$\begin{aligned} \partial_{z_i} f\,(T \circ z) &= \partial_{z_i} f\,(z') \\ &= \frac{\partial z'_j}{\partial z_i} \frac{\partial}{\partial z'_j} f\,(z') \\ &= S_{ji} \partial_{z'_j} f\,(z') \end{aligned} \tag{5.21}$$

Therefore,

$$\begin{aligned} \left(\partial_{z_i}\,(T \circ f)\right) \left(\partial_{z_j}\,(T \circ g)\right) \Omega^C_{ij} &= \left(\partial_{z'_k} f\,(z')\right) S_{ki} S_{lj} \left(\partial_{z'_l} g\,(z')\right) \Omega^C_{ij} \\ &= \left(\partial_{z'_k} f\,(z')\right) \left(\partial_{z'_l} g\,(z')\right) S_{ki} \Omega^C_{ij} S^T_{jl} \end{aligned} \tag{5.22}$$

Thus, expanding eq. (5.13) using eq. (5.20) yields simply

$$\hbar S_{ki} \Omega^C_{ij} S^T_{jl} = \hbar' \Omega^{NC}_{kl}, \tag{5.23}$$

or, in matrix form,

$$\hbar S \Omega^C S^T = \hbar' \Omega^{NC}. \tag{5.24}$$

Note that $\hbar$ and $\hbar'$ appear since the argument of the exponential of the commutative Moyal product is

$$\frac{i\hbar}{2} \overleftarrow{\partial_{z_i}} \Omega^{NC}_{ij} \overrightarrow{\partial_{z_j}}, \tag{5.25}$$

and the noncommutative analogous is

$$\frac{i\hbar'}{2}\overleftarrow{\partial_{z_i}}\Omega^{NC}_{ij}\overrightarrow{\partial_{z_j}}. \tag{5.26}$$

Replacing eqs. (5.15) and (5.19) into eq. (5.24), we get:

$$\begin{aligned}
\begin{pmatrix} \frac{1}{\hbar}\Theta & \frac{\hbar'}{\hbar}\mathrm{Id}_{d\times d} \\ \frac{\hbar'}{\hbar}\mathrm{Id}_{d\times d} & \frac{1}{\hbar}N \end{pmatrix} &= \begin{pmatrix} \boldsymbol{\alpha} & \boldsymbol{\beta} \\ \boldsymbol{\gamma} & \boldsymbol{\zeta} \end{pmatrix}\begin{pmatrix} 0 & \mathrm{Id}_{d\times d} \\ -\mathrm{Id}_{d\times d} & 0 \end{pmatrix}\begin{pmatrix} \boldsymbol{\alpha}^T & \boldsymbol{\gamma}^T \\ \boldsymbol{\beta}^T & \boldsymbol{\zeta}^T \end{pmatrix} \\
&= \begin{pmatrix} \boldsymbol{\alpha} & \boldsymbol{\beta} \\ \boldsymbol{\gamma} & \boldsymbol{\zeta} \end{pmatrix}\begin{pmatrix} \boldsymbol{\beta}^T & \boldsymbol{\zeta}^T \\ -\boldsymbol{\alpha}^T & -\boldsymbol{\gamma}^T \end{pmatrix} \\
&= \begin{pmatrix} \boldsymbol{\alpha}\boldsymbol{\beta}^T - \boldsymbol{\beta}\boldsymbol{\alpha}^T & \boldsymbol{\alpha}\boldsymbol{\zeta}^T - \boldsymbol{\beta}\boldsymbol{\gamma}^T \\ \boldsymbol{\gamma}\boldsymbol{\beta}^T - \boldsymbol{\zeta}\boldsymbol{\alpha}^T & \boldsymbol{\gamma}\boldsymbol{\zeta}^T - \boldsymbol{\zeta}\boldsymbol{\gamma}^T \end{pmatrix}.
\end{aligned} \tag{5.27}$$

But these are exactly the equations we obtained when developing the Darboux transform in Chapter 3 (eqs. (3.21), (3.22) and (3.23)). Therefore, the required transformation is a Darboux transformation, and therefore the transformation:

$$W^{-1}\left(D \circ W_{NC}\left[*\right]\right) \tag{5.28}$$

relates operators in NCQM with operators in QM and respects eq. (5.12).

5.3 Density Matrix in NCQM and QM

Now, since we wanted to study states in NCQM, and since the density matrix $\widehat{\rho}_\psi$ associated with a state $|\psi\rangle$ is an operator (i.e. $\widehat{\rho}_\psi = |\psi\rangle\langle\psi|$), we can use the transformation (5.1) to relate this density matrix in NCQM to an operator $\widehat{M}$ in QM:

$$\widehat{M} = W^{-1}\left(D \circ W_{NC}\left[\widehat{\rho}^{NC}_\psi\right]\right). \tag{5.29}$$

The objective now is to prove that this operator behaves as density matrix $\widehat{M} = \widehat{\rho} = |\psi'\rangle\langle\psi'|$.

Properties:

1. $\left\langle\widehat{A}\right\rangle = \mathrm{Tr}\left(\widehat{M}\widehat{\widehat{A}}\right)$ for any operator $\widehat{A}$, and where $\widehat{\widehat{A}} = W^{-1}\left(D \circ W_{NC}\left[\hat{A}\right]\right)$.

2. $\mathrm{Tr}\left(\widehat{M}\right) = 1$ (normalization)

3. $\widehat{M}^\dagger = \widehat{M}$ (hermicity)

4. $\mathrm{Tr}\left(\widehat{M}^2\right) = 1$ (purity)

Proof of 1.:

$$\mathrm{Tr}\left(\widehat{M}\widehat{\widetilde{A}}\right) = \int d^{2D}z\, W\left(\widehat{M}\widehat{\widetilde{A}}\right) \tag{5.30}$$

because of eq. (2.18). Because of the property (2.9) of W ,

$$\begin{aligned}\int d^{2d}z\, W\left(\widehat{M}\widehat{\widetilde{A}}\right) &= \int d^{2d}z\, W\left(\widehat{M}\right) \star_C W\left(\widehat{\widetilde{A}}\right) \\ &= \int d^{2d}z \left(D \circ W_{NC}\left[\widehat{\rho}_\psi^{NC}\right]\right) \star_C \left(D \circ W_{NC}\left[\widehat{A}\right]\right),\end{aligned} \tag{5.31}$$

where we used the definition of $\widehat{M}$ and $\widehat{\widetilde{A}}$, and that $W\left(W^{-1}[*]\right) = Id$. Then,

$$\begin{aligned}\mathrm{Tr}\left(\widehat{M}\widehat{\widetilde{A}}\right) &= \int d^{2d}z \left(D \circ W_{NC}\left[\widehat{\rho}_\psi^{NC}\right]\right) \star_C \left(D \circ W_{NC}\left[\widehat{A}\right]\right) \\ &= \int d^{2d}z\, D \circ \left(W_{NC}\left[\widehat{\rho}_\psi^{NC}\right] \star_{NC} W_{NC}\left[\widehat{A}\right]\right),\end{aligned} \tag{5.32}$$

as we saw in the previous section. Since D is a change of variables, we can incorporate it into the integration, and thus

$$\begin{aligned}\mathrm{Tr}\left(\widehat{M}\widehat{\widetilde{A}}\right) &= \int d^{2d}z\, W_{NC}\left[\widehat{\rho}_\psi^{NC}\right] \star_{NC} W_{NC}\left[\widehat{A}\right] \\ &= \int d^{2d}z\, W_{NC}\left[\widehat{\rho}_\psi^{NC}\widehat{A}\right] \\ &= \mathrm{Tr}\left(\widehat{\rho}_\psi^{NC}\widehat{A}\right) = \left\langle\widehat{A}\right\rangle = \left\langle\widehat{\widetilde{A}}\right\rangle,\end{aligned} \tag{5.33}$$

where we used eqs. (3.3) and (3.4).

Note that $\langle A\rangle = \left\langle\widetilde{A}\right\rangle$ because the result of a measurement cannot depend on the way one chooses to represent the operators.

Proof of 2.: Similarly to 1),

$$\begin{aligned}\mathrm{Tr}\left(\widehat{M}\right) &= \int d^{2d}z\, W\left(\widehat{M}\right)\\ &= \int d^{2d}z\, \left(D\circ W_{NC}\left[\widehat{\rho}_{\psi}^{NC}\right]\right),\end{aligned}\tag{5.34}$$

by definition of $\widehat{M}$ and using eq. (2.18). Again, since D is a change of variables, we can incorporate it into the integration, and thus

$$\begin{aligned}\mathrm{Tr}\left(\widehat{M}\right) &= \int d^{2d}z\, W_{NC}\left[\widehat{\rho}_{\psi}^{NC}\right]\\ &= \mathrm{Tr}\left(\widehat{\rho}_{\psi}^{NC}\right) = 1.\end{aligned}\tag{5.35}$$

Proof of 3.: By definition of $\widehat{M}$,

$$\begin{aligned}\widehat{M}^{\dagger} &= \left[W^{-1}\left(D\circ W_{NC}\left[\widehat{\rho}_{\psi}^{NC}\right]\right)\right]^{\dagger}\\ &= \left[\int \frac{d^{2d}k}{(2\pi)^{2d}}\int d^{2d}z\, \left(D\circ W_{NC}\left[\widehat{\rho}_{\psi}^{NC}\right]\right)^{*}\cdot e^{ik_i\hat{z}^i}e^{-ik_iz^i}\right]^{\dagger}\\ &= \int \frac{d^{2d}k}{(2\pi)^{2d}}\int d^{2d}z\, \left(D\circ W_{NC}\left[\widehat{\rho}_{\psi}^{NC}\right]\right)^{*}\cdot e^{-ik_i\hat{z}^i}e^{ik_iz^i},\end{aligned}\tag{5.36}$$

where we used the explicit formula for the Weyl transform (see eq. (2.7)) in the second line. Note that we also used in the last line the fact that only $e^{ik_i\hat{z}^i}$ is an operator, and that position and momenta operators in QM are hermitian. Changing the integral variables via the transformation $z_i \to -z_i$, we get

$$\begin{aligned}\widehat{M}^{\dagger} &= \int \frac{d^{2d}k}{(2\pi)^{2d}}\int d^{2d}z\, \left(D\circ W_{NC}\left[\widehat{\rho}_{\psi}^{NC}\right]\right)^{*}\cdot e^{-ik_i\hat{z}^i}e^{ik_iz^i}\\ &= (-1)^{2d}\int \frac{d^{2d}k}{(2\pi)^{2d}}\int d^{2d}z\, \left(D\circ W_{NC}\left[\widehat{\rho}_{\psi}^{NC}\right]\right)^{*}\cdot e^{ik_i\hat{z}^i}e^{-ik_iz^i}\\ &= W^{-1}\left(\left(D\circ W_{NC}\left[\widehat{\rho}_{\psi}^{NC}\right]\right)^{*}\right).\end{aligned}\tag{5.37}$$

Because the Darboux transformation is real, we have

$$\left(D\circ W_{NC}\left[\widehat{\rho}_{\psi}^{NC}\right]\right)^{*} = D\circ\left(W_{NC}\left[\widehat{\rho}_{\psi}^{NC}\right]\right)^{*}.\tag{5.38}$$

Because $\widehat{\rho}_\psi^{NC}$ is hermitian,

$$\begin{aligned} W_{NC}\left[\widehat{\rho}_\psi^{NC}\right] &= W_{NC}\left[\widehat{\rho}_\psi^{NC,\dagger}\right] \\ &= \left(W_{NC}\left[\widehat{\rho}_\psi^{NC}\right]\right)^*, \end{aligned} \tag{5.39}$$

and thus

$$\begin{aligned} \widehat{M}^\dagger &= W^{-1}\left(\left(D\circ W_{NC}\left[\widehat{\rho}_\psi^{NC}\right]\right)^*\right) \\ &= W^{-1}\left(D\circ W_{NC}\left[\widehat{\rho}_\psi^{NC}\right]\right) \\ &= \widehat{M}. \end{aligned} \tag{5.40}$$

Proof of 4.: As before,

$$\begin{aligned} \mathrm{Tr}\left(\widehat{M}^2\right) &= \int d^{2d}z\, W\left(\widehat{M}^2\right) \\ &= \int d^{2d}z\, \left(D\circ W_{NC}\left[\widehat{\rho}_\psi^{NC}\right]\right)\star_C\left(D\circ W_{NC}\left[\widehat{\rho}_\psi^{NC}\right]\right) \\ &= \int d^{2d}z\, D\circ\left(W_{NC}\left[\widehat{\rho}_\psi^{NC}\right]\star_{NC} W_{NC}\left[\widehat{\rho}_\psi^{NC}\right]\right) \\ &= \int d^{2d}z\, W_{NC}\left[\widehat{\rho}_\psi^{NC}\right]\star_{NC} W_{NC}\left[\widehat{\rho}_\psi^{NC}\right] \\ &= \int d^{2d}z\, W_{NC}\left[\widehat{\rho}_\psi^{NC}\widehat{\rho}_\psi^{NC}\right] \\ &= \mathrm{Tr}\left(\widehat{\rho}_\psi^{NC}\widehat{\rho}_\psi^{NC}\right)=1 \end{aligned}$$

Note that this is exactly Property 1 with $\widehat{\widehat{A}}=\widehat{M}$.

Since $\widehat{M}$ obeys the properties above, it behaves as density matrix associated with some state $|\psi'\rangle$ in QM, i.e.

$$\widehat{M}=\widehat{\rho}=|\psi'\rangle\langle\psi'|\,. \tag{5.41}$$

Furthermore, because of property 1.,

$$E=\mathrm{Tr}\left(\widehat{\rho}_\psi^{NC}\widehat{H}_{NC}\right)=\mathrm{Tr}\left(\widehat{\rho}\widehat{H}_C\right), \tag{5.42}$$

where $\widehat{H}_C = W^{-1}\left(D \circ W_{NC}\left[\widehat{H}_{NC}\right]\right)$ is corresponding the Hamiltonian in QM.

We also obtain

$$|\langle\psi_{NC}|\phi_{NC}\rangle|^2 = \mathrm{Tr}\left(\widehat{\rho}_\psi^{NC}\widehat{\rho}_\phi^{NC}\right) = \mathrm{Tr}\left(\widehat{\rho}_\psi\widehat{\rho}_\phi\right) = |\langle\psi|\phi\rangle|^2\,, \tag{5.43}$$

and thus the orthogonality of states is preserved.

Hence, we obtain a one-to-one correspondence between states in NCQM and states in QM.

5.4 Transformation of $f\left(\widehat{Q}_{NC}, \widehat{P}_{NC}\right)$ operators

In this section, we will show the transformation of $f\left(\widehat{Q}_{NC}, \widehat{P}_{NC}\right)$ operators by eq. (5.28) for the W_{NC} map in Ref. [14].

Consider the NC Wigner Transform in eq. (3.2):

$$W_{NC}\left(\hat{A}\right) = \hbar^{-d}\int d^dx\, d^dy\, e^{-iP^C(z)\cdot y}\delta\left(x - Q^C(z)\right)\left\langle x + \frac{\hbar}{2}y|\hat{A}|x - \frac{\hbar}{2}y\right\rangle_{Q^C},$$

with the Darboux transformation (see eqs. (3.25) and (3.26)):

$$\begin{aligned} Q_i &= Q_i^C + \frac{\theta_{ij}}{2\hbar}P_j^C\,, \\ P_i &= P_i^C - \frac{\eta_{ij}}{2\hbar}Q_j^C\,, \end{aligned}$$

where Q_i^C and P_i^C are commutative position and momenta variables and Q_i and P_i are the noncommutative variables.

For $\theta, \eta \ll \hbar$, $\hbar' \approx \hbar$ and this transformation is easily invertible.

$$Q_i^C = Q_i - \frac{\theta_{ij}}{2\hbar}P_j \tag{5.44}$$

$$P_i^C = P_i + \frac{\eta_{ij}}{2\hbar}Q_j. \tag{5.45}$$

Thus,

$$W_{NC}\left(\widehat{Q}_i\right) = h^{-d}\int d^dx\, d^dy\, e^{-iP^C\cdot y}\delta\left(x - Q^C\right)\left\langle x + \frac{\hbar}{2}y|\widehat{Q}_i|x - \frac{\hbar}{2}y\right\rangle_{\widehat{Q^C}}$$

$$= h^{-d} \int d^d x\, d^d y\, e^{-iP^C \cdot y} \delta\left(x - Q^C\right) \left\langle x + \frac{\hbar}{2} y | \widehat{Q^C}_i + \frac{\theta_{ij}}{2\hbar} \widehat{P^C}_j | x - \frac{\hbar}{2} y \right\rangle_{\widehat{Q^C}}$$

$$= h^{-d} \int d^d x\, d^d y\, e^{-iP^C \cdot y} \delta\left(x - Q^C\right) \left(x_i \delta\left(\frac{\hbar}{2} y\right) + \frac{\theta_{ij}}{2\hbar} \frac{d}{dQ_j^C} \delta\left(\frac{\hbar}{2} y\right) \right). \tag{5.46}$$

Integrating by parts on the second term, and having boundary term vanish, we get

$$\begin{aligned} W_{NC}\left(\widehat{Q}_i\right) &= \hbar^{-d} \int d^d x\, d^d y\, e^{-iP^C \cdot y} \delta\left(x - Q^C(z)\right) \left(x_i \delta\left(\frac{\hbar}{2} y\right) + \frac{\theta_{ij}}{2\hbar} \delta\left(\frac{\hbar}{2} y\right) P_j^C \right) \\ &= \int d^d x\, \delta\left(x - Q^C(z)\right) \left(x_i + \frac{\theta_{ij}}{2\hbar} P_j^C \right) \\ &= \left(Q_i^C + \frac{\theta_{ij}}{2\hbar} P_j^C \right) = Q_i, \end{aligned} \tag{5.47}$$

as expected from a NC Wigner transform.

Then, using eq. (2.7), the transformation (5.28) becomes:

$$W^{-1}\left(D \circ W_{NC}\left[\widehat{Q}_i\right]\right) = W^{-1}\left(D \circ Q_i\right) = W^{-1}\left(Q_i^C + \frac{\theta_{ij}}{2\hbar} P_j^C \right) =$$

$$= \int \frac{d^{2d}k\, d^{2d}k’}{(2\pi)^{2d}} \int d^d Q^C\, d^d P^C \left(Q_i^C + \frac{\theta_{ij}}{2\hbar} P_j^C \right) \cdot e^{ik_l \widehat{Q_i^C} + ik_l' \widehat{P_i^C}} e^{-ik_l Q_i^C - ik_l' P_i^C}. \tag{5.48}$$

The integration of each term is done by simple integration by parts of type $\int dx\, xe^{-x}$. For the first term:

$$\int \frac{d^{2d}k\, d^{2d}k’}{(2\pi)^{2d}} \int d^d Q^C\, d^d P^C\, Q_i^C \cdot e^{ik_l \widehat{Q_i^C} + ik_l' \widehat{P_i^C}} e^{-ik_l Q_i^C - ik_l' P_i^C} =$$

$$= \int \frac{d^{2d}k\, d^{2d}k’}{(2\pi)^{2d}} e^{ik_l \widehat{Q_i^C} + ik_l' \widehat{P_i^C}} \int d^d P^C\, k_i \left(-xe^{-x} - e^{-x}\right)|_{-\infty}^{\infty} e^{-ik_l' P_i^C} =$$

$$= \int \frac{d^{2d}k\, d^{2d}k’}{(2\pi)^{2d}} k_i e^{ik_l \widehat{Q_i^C} + ik_l' \widehat{P_i^C}} \int d^d P^C\, e^{-ik_l' P_i^C} =$$

$$= \int \frac{d^{2d}k\, d^{2d}k’}{(2\pi)^{2d}} k_i e^{ik_l \widehat{Q_i^C} + ik_l' \widehat{P_i^C}} = \widehat{Q_i^C}. \tag{5.49}$$

Similarly for the second term,

$$\int \frac{d^{2d}k\, d^{2d}k'}{(2\pi)^{2d}} \int d^dQ^C\, d^dP^C\, P_i^C \cdot e^{ik_l\widehat{Q_i^C}+ik'_l\widehat{P_i^C}}\, e^{-ik_lQ_i^C-ik'_lP_i^C} = \widehat{P_i^C}. \tag{5.50}$$

Thus,

$$W^{-1}\left(D\circ W_{NC}\left[\widehat{Q}_i\right]\right) = \widehat{Q_i^C} + \frac{\theta_{ij}}{2\hbar}\widehat{P_j^C}. \tag{5.51}$$

Note that these operators obey the Heisemberg algebra by definition (see Ref. [7]). The same process can be repeated for $W^{-1}\left(D\circ W_{NC}\left[\widehat{P}_i\right]\right)$, yielding:

$$\begin{aligned} W_{NC}\left(\widehat{P}_i\right) &= \hbar^{-d}\int d^dx\, d^dy\, e^{-iP^C\cdot y}\delta\left(x-Q^C\right)\left\langle x+\frac{\hbar}{2}y|\widehat{P}_i|x-\frac{\hbar}{2}y\right\rangle_{\widehat{Q^C}} \\ &= P_i, \end{aligned} \tag{5.52}$$

and thus

$$W^{-1}\left(D\circ W_{NC}\left[\widehat{P}_i\right]\right) = W^{-1}\left(D\circ P_i\right) =$$

$$= W^{-1}\left(P_i^C - \frac{\eta_{ij}}{2\hbar}Q_j^C\right) = \widehat{P_i^C} - \frac{\eta_{ij}}{2\hbar}\widehat{Q_i^C}. \tag{5.53}$$

Note that this is simply the Darboux transformation of the operators, which is what is expected. However, for a general function, $f\left(\widehat{Q},\widehat{P}\right)$, this might not be true.

In the case

$$f\left(\widehat{Q},\widehat{P}\right) = \sum_{n,m,i,j}\alpha_{nm}\widehat{Q}_i^n\widehat{P}_j^m, \tag{5.54}$$

we have

$$W_{NC}\left(f\left(\widehat{Q},\widehat{P}\right)\right) =$$

$$= \sum_{n,m,i,j}\alpha_{nmij}h^{-d}\int d^dx\, d^dy\, e^{-iP^C\cdot y}\delta\left(x-Q^C\right)\left\langle x+\frac{\hbar}{2}y|\widehat{Q}_i^n\widehat{P}_j^m|x-\frac{\hbar}{2}y\right\rangle_{\widehat{Q^C}}$$

$$= \sum_{n,m,i,j}\alpha_{nmij}\underbrace{Q_i\star_{NC}\ldots\star_{NC}Q_i}_{n\,times}\star_{NC}\underbrace{P_j\star_{NC}\ldots\star_{NC}P_j}_{m\,times}$$

$$= \sum_{n,m,i,j} \alpha_{nmij} Q_i^n \star_{NC} P_j^m. \tag{5.55}$$

However, because of eq. (5.13),

$$D \circ W_{NC} \left(f \left(\widehat{Q}, \widehat{P} \right) \right) = \sum_{n,m,i,j} \alpha_{nmij} \left(Q_i^C + \frac{\theta_{ik}}{2\hbar} P_k^C \right)^n \star_C \left(P_j^C - \frac{\eta_{jl}}{2\hbar} Q_l^C \right)^m, \tag{5.56}$$

and thus

$$\begin{aligned} W^{-1} \left(D \circ W_{NC} \left[f \left(\widehat{Q}, \widehat{P} \right) \right] \right) &= \sum_{n,m,i,j} \alpha_{nmij} \left(\widehat{Q_i^C} + \frac{\theta_{ik}}{2\hbar} \widehat{P_k^C} \right)^n \left(\widehat{P_j^C} - \frac{\eta_{jl}}{2\hbar} \widehat{Q_l^C} \right)^m \\ &= f \left(\widehat{Q_i^C} + \frac{\theta_{ik}}{2\hbar} \widehat{P_k^C}, \widehat{P_j^C} - \frac{\eta_{jl}}{2\hbar} \widehat{Q_l^C} \right). \end{aligned} \tag{5.57}$$

where we used eq. (2.9).

Note that this is simply the Darboux transformation of the operators, which is unsurprising since this particular NC Wigner Transform was built to obey the map (see Ref. [14]):

$$\begin{array}{ccc} \widehat{A}\left(\widehat{\xi}\right) & \xrightarrow{\widehat{D}} & \widehat{A}'(z) = \widehat{A}\left(\widehat{\xi}\left(\widehat{z}\right)\right) \\ W_\xi \downarrow & & \downarrow W_z^\xi \\ A(\xi) & \xrightarrow[D]{} & A'(z) = A(\xi(z)) \end{array}$$

where $\widehat{A}$ denote operators and A denotes phase-space functions, and ξ and z denote positions and momenta related by a Darboux transform. This property was what allowed eq. (5.55).

While this might seem redundant, if one instead of working with operators in NCQM works only with functions in phase-space (for example, using Moyal's equation, eq. (2.17)), one needs only do the transformation

$$\widehat{F} = W^{-1} \left(D \circ f(z) \right), \tag{5.58}$$

for a given phase-space function $f(z)$, to obtain a corresponding operator in QM.

Furthermore, this allows us to understand the acting of operators on state. In particular, since in NCQM position and momenta operators do not commute, they do not form a Complete Set of Commuting Observables (CSCO) and thus one cannot define a position basis or a momentum basis. However, by using operators in QM, this can be done directly, since in QM these operators do form a CSCO, which was the motivation for the previous section.

Chapter 6

No-Go Theorems in Noncommutative Quantum Mechanics

Having established a transformation between operators and states in NCQM and QM, one can prove that No-Go Theorems valid in QM also hold in NCQM. The idea is to relate the proof for QM, use the transformation (5.1) to set equivalent theorems in NCQM.

6.1 No-Cloning in Noncommutative Quantum Mechanics

The outline of the proof is simple. Use the same assumptions as in QM, use the relation between NCQM and QM and the fact that the corresponding operator to $\widehat{U}_{NC}$ is also unitary to show that if cloning were possible, $\langle\phi|\psi\rangle_{NC} = \langle\phi|\psi\rangle^2_{NC}$ for any states $|\phi\rangle_{NC}$ and $|\psi\rangle_{NC}$.

Theorem (No-Cloning in NCQM): Let $|\psi\rangle_{NC}$ be a generic noncommutative quantum state and $|0\rangle_{NC}$ be an empty state. Then it is not possible evolve to these two states into two copies of $|\psi\rangle_{NC}$ for any quantum state $|\psi\rangle_{NC}$.

Proof: Assume that cloning is possible in NCQM. Then there is a Hamiltonian, $\widehat{H}_{NC}$, so that

$$\widehat{U}_{NC} \, |\psi\rangle_{A,NC} \, |0\rangle_{B,NC} = |\psi\rangle_{A,NC} \, |\psi\rangle_{B,NC} \tag{6.1}$$

for any given state $|\psi\rangle_{NC}$ and an empty state $|0\rangle_{NC}$ and where $\widehat{U}_{NC}$ is the time evolution operator associated with $\widehat{H}_{NC}$.

Then, if $|\phi\rangle_{NC}$ is another state, i.e.,

$$\widehat{U}_{NC}\,|\phi\rangle_{A,NC}\,|0\rangle_{B,NC} = |\phi\rangle_{A,NC}\,|\phi\rangle_{B,NC}\,, \tag{6.2}$$

one has:

$$\begin{aligned} \langle\phi|\psi\rangle_{NC} &= \left(\mathrm{Tr}\left(\rho_\phi^{NC}\rho_\psi^{NC}\right)\right)^{\frac{1}{2}} \\ &= \left(\mathrm{Tr}\left(\rho_{\phi'}\rho_{\psi'}\right)\right)^{\frac{1}{2}} \\ &= e^{i\alpha}\,\langle\phi'|\psi'\rangle\,, \end{aligned} \tag{6.3}$$

where α is a real number and where we used eq. (5.43).

Now we will prove that if $\widehat{U}_{NC}$ is unitary, $\widehat{V}$, defined as (c. f. eq. (5.28))

$$\widehat{V} = W^{-1}\left(D\circ W_{NC}\left[\widehat{U}_{NC}\right]\right), \tag{6.4}$$

is also unitary, and vice-versa.

Proof: Taking

$$\widehat{V} = W^{-1}\left(D\circ W_{NC}\left[\widehat{U}_{NC}^\dagger\right]\right), \tag{6.5}$$

we have

$$\begin{aligned} \widehat{V}^\dagger\widehat{V} &= W^{-1}\left(D\circ W_{NC}\left[\widehat{U}_{NC}^\dagger\right]\right)\cdot W^{-1}\left(D\circ W_{NC}\left[\widehat{U}_{NC}\right]\right) \\ &= W^{-1}\left(\left(D\circ W_{NC}\left[\widehat{U}_{NC}^\dagger\right]\right)\star_C\left(D\circ W_{NC}\left[\widehat{U}_{NC}\right]\right)\right), \end{aligned} \tag{6.6}$$

by eq. (2.9). Using eq. (5.12), one also has:

$$\begin{aligned} \widehat{V}^\dagger\widehat{V} &= W^{-1}\left(D\circ\left(W_{NC}\left[\widehat{U}_{NC}^\dagger\right]\star_{NC}W_{NC}\left[\widehat{U}_{NC}\right]\right)\right) \\ &= W^{-1}\left(D\circ\left(W_{NC}\left[\widehat{U}_{NC}^\dagger\widehat{U}_{NC}\right]\right)\right), \end{aligned} \tag{6.7}$$

where we used eq. (3.3) in the last step.

Since both W_{NC} and W map the Identity to 1,

$$\widehat{V}^\dagger\widehat{V} = \mathrm{Id}_{d\times d} \Leftrightarrow \widehat{U}_{NC}^\dagger\widehat{U}_{NC} = \mathrm{Id}_{d\times d}. \tag{6.8}$$

Given that $\widehat{V}$ is also the time evolution operator associated to a Hamiltonian, $\widehat{\mathcal{H}}$, defined as:

$$\widehat{\mathcal{H}} = W^{-1}\left(D \circ W_{NC}\left[\widehat{H}_{NC}\right]\right), \tag{6.9}$$

one has:

$$\widehat{V}\,|\psi'\rangle_A\,|0\rangle_B = |\psi'\rangle_A\,|\psi'\rangle_B\,, \tag{6.10}$$

$$\widehat{V}\,|\phi'\rangle_A\,|0\rangle_B = |\phi'\rangle_A\,|\phi'\rangle_B\,, \tag{6.11}$$

and therefore:

$$\begin{aligned}
\langle\phi|\psi\rangle_{NC} &= e^{i\alpha}\,\langle\phi'|\psi'\rangle \\
&= e^{i\alpha}\,\langle\phi'|\psi'\rangle_A\,\langle 0|0\rangle_B \\
&= e^{i\alpha}\left(\langle\phi'|_A\,\langle 0|_B\right)\widehat{V}^\dagger\widehat{V}\left(|\psi'\rangle_A\,|0\rangle_B\right) \\
&= e^{i\alpha}\,\langle\phi'|\psi'\rangle^2 \\
&= e^{-i\alpha}\,\langle\phi|\psi\rangle_{NC}^2\,,
\end{aligned} \tag{6.12}$$

from eq. (6.3), and thus $\langle\phi|\psi\rangle_{NC} = 0$ or $\langle\phi|\psi\rangle_{NC} = e^{i\alpha}$, which cannot be true for all states $|\phi\rangle_{NC}$ and $|\psi\rangle_{NC}$. The contradiction arises from the assumption that cloning was possible for any given state. Hence, likewise in QM, there can be no cloning in NCQM.

Likewise in QM, the proof can be performed in reversed order, and thus the No-Deleting Theorem is also valid in NCQM.

6.2 No-Go Generalization in Noncommutative Quantum Mechanics

The proof is similar to the one in QM. The cases where there are no superposition are the No-Cloning and No-Deleting Theorems. Then, using the linearity of the Wigner-Weyl transformation and its preservation of unitarity, we can prove the general case.

Theorem: There is no Hamiltonian, $\widehat{H}_{NC}$, with an associated time evolution operator $\widehat{U}_{NC}$ so that for a fixed state $|\phi\rangle_{NC}$ and for any state $|\psi\rangle_{NC}$ the following is true:

$$\widehat{U}_{NC}\,|\psi\rangle_{NC}^{\otimes k}\,|0\rangle_{NC}^{\otimes N-k} = |\varphi\rangle_{NC}^{\otimes n}\,|0\rangle_{NC}^{\otimes N-n}\,, \tag{6.13}$$

where $|\varphi\rangle_{NC} = \alpha\,|\psi\rangle_{NC} + \beta\,|\phi\rangle_{NC}$, with $|\alpha|^2 + |\beta|^2 = 1$, and where we used the notation:

$$|\psi\rangle^{\otimes k} = \underbrace{|\psi\rangle \otimes \ldots \otimes |\psi\rangle}_{k\,times}. \tag{6.14}$$

Proof: First, note that as in QM, when $\beta = 0$ (i.e. there is no superposition with another state), if $k < n$, we have the No-Cloning Theorem, and if $k > n$, we have the No-Deleting Theorem. Thus, we only need to prove the case $0 < \beta < 1$.

Suppose there is a Hamiltonian, $\widehat{H}_{NC}$, so that

$$\widehat{U}_{NC}\,|\psi\rangle_{NC}^{\otimes k}\,|0\rangle_{NC}^{\otimes N-k} = |\varphi\rangle_{NC}^{\otimes n}\,|0\rangle_{NC}^{\otimes N-n} \tag{6.15}$$

for a state $|\psi\rangle_{NC}$ and with $|\varphi\rangle_{NC} = \alpha\,|\psi\rangle_{NC} + \beta\,|\phi\rangle_{NC}$.

Then, if we instead used the state $e^{i\theta}\,|\psi\rangle$, then

$$\widehat{U}_{NC}e^{ik\theta}\,|\psi\rangle_{NC}^{\otimes k}\,|0\rangle_{NC}^{\otimes N-k} = |\varphi'\rangle_{NC}^{\otimes n}\,|0\rangle_{NC}^{\otimes N-n}\,, \tag{6.16}$$

where $|\varphi'\rangle_{NC} = \alpha e^{i\theta}\,|\psi\rangle_{NC} + \beta\,|\phi\rangle_{NC}$.

However, because $|\psi\rangle_{NC} \propto e^{i\theta}\,|\psi\rangle_{NC}$, assuming that the states are normalized, taking the hermitian conjugate of eq. (6.15) and multiplying it to eq. (6.16) yields:

$$e^{ik\theta} = \langle\varphi|\varphi'\rangle_{NC}^{n}\,. \tag{6.17}$$

Using the relation between NCQM and QM, as before:

$$\begin{aligned} e^{i\theta\frac{k}{n}} &= \langle\varphi|\varphi'\rangle_{NC} \\ &= \left(\mathrm{Tr}\left(\rho_{\varphi}^{NC}\rho_{\varphi'}^{NC}\right)\right)^{\frac{1}{2}} \\ &= \left(\mathrm{Tr}\left(\rho_{\varphi}\rho_{\varphi'}\right)\right)^{\frac{1}{2}} \\ &= e^{i\alpha}\,\langle\varphi|\varphi'\rangle\,. \end{aligned} \tag{6.18}$$

where we have defined

$$\rho_{\varphi} = W^{-1}\left(D \circ W_{NC}\left[\rho_{\varphi}^{NC}\right]\right). \tag{6.19}$$

Thus, from the linearity of the Weyl and Wigner transform, we have:

$$|\varphi\rangle = \alpha\,|\psi\rangle + \beta\,|\phi\rangle \tag{6.20}$$

$$|\varphi'\rangle = \alpha e^{i\theta}\,|\psi\rangle + \beta\,|\phi\rangle\,, \tag{6.21}$$

where $|\psi\rangle$ and $|\phi\rangle$ are the state representatives of

$$\rho_\psi = W^{-1}\left(D \circ W_{NC}\left[\rho_\psi^{NC}\right]\right), \tag{6.22}$$

$$\rho_\phi = W^{-1}\left(D \circ W_{NC}\left[\rho_\phi^{NC}\right]\right). \tag{6.23}$$

Thus, we must have

$$\langle\varphi_C|\varphi'_C\rangle \propto e^{i\theta}\,|\alpha|^2 + |\beta|^2\,, \tag{6.24}$$

and since $|\alpha|^2$ and $|\beta|^2$ are real numbers, eq. (6.18) can only be true if $\beta = 0$, which is a contradiction to the hypothesis $0 < \beta < 1$. This concludes the proof for NCQM.

Chapter 7

Conclusions

In this work, No-Go Theorems in the context of NCQM are addressed. After a brief review of the Wigner-Weyl formalism of QM and NCQM, the No-Cloning Theorem and also a generalization of the No-Go Theorems has been discussed in the context of QM.

By establishing a relation between NCQM and QM, it was proven that these theorems still hold when one considers a phase space with deformed commutation relations. This is mostly due to the fact that unitariety and linearity, key features in QM, are preserved in the NCQM Wigner-Weyl formalism, and therefore most of the results derived from linearity should still be true in NCQM.

Furthermore, the fact that phase-space integrations are invariant under different Moyal products also contributed for these results to hold regardless of the commutation relations between phase space variables considered, and is a key factor in the evaluation of expectation values, as they can be calculated independently of the way one chooses to represent the operators.

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